Chew-Low Formalism for Two Interactions*

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An extension of the method of Chew and Low is developed for interacting boson-fermion systems for which a portion (H_1) of the interaction can be treated exactly and is included in the zero-order problem. It is applied to the scalar pair term of meson theory.

INTRODUCTION

ANALYSES of low-energy pion processes on the basis of the approximation method of Chew and Low[1] have proved valuable in correlating pseudoscalar meson theory with experiment. It is a new feature of this development that one deals directly with physical nucleon states (and renormalized coupling constants).

In this note we report an extension of these methods to interacting systems for which a portion of the interaction can be treated "exactly" and is included in the zero-order problem. A simple example of this in ordinary potential scattering is the analysis of proton-proton scattering in terms of Coulomb wave functions.

In potential scattering with $H_T=H_1+H_2+H_0$, the S matrix can be given in two equivalent forms

$$\begin{aligned}S_{ba}&=\delta_{ba}-2\pi i\delta(E_b-E_a)\langle\Psi_b^{(-)}|H_1+H_2|\chi_a\rangle\\&=\langle\varphi_b^{(-)}|\varphi_a^{(+)}\rangle-2\pi i\delta(E_b-E_a)\\&\qquad\times\langle\Psi_b^{(-)}|H_2|\varphi_a^{(+)}\rangle,\end{aligned}\tag{1}$$

with

$$\begin{aligned}H_T\Psi_b^{(\pm)}&=E_b\Psi_b^{(\pm)},\\(H_1+H_0)\varphi_a^{(\pm)}&=E_a\varphi_a^{(\pm)},\\H_0\chi_a&=E_a\chi_a.\end{aligned}\tag{2}$$

The central result contained in this paper is the field-theoretic analog of Eq. (1), in which there appears only physical nucleon states, i.e., eigenstates of the total Hamiltonian, H_T.

An application of this method to the scalar pair term of meson field theory is discussed and compared with earlier work.[2]

DEVELOPMENT

Consider first a Hamiltonian

$$\begin{aligned}H&=H_1+H_0,\\H_0&=\textstyle\sum_k\omega_k a_k^\dagger a_k.\end{aligned}\tag{3}$$

The operators a_k and $a_k^\dagger$ destroy and create mesons in free particle states specified by quantum numbers k. Let φ_0, $\varphi_k^{(\pm)}$, $\cdots$ be the complete set of eigenstates of H, φ_0 corresponding to the ground state, i.e., a physical nucleon, $\varphi_k^{(\pm)}$ to a one-meson scattering state with outgoing $(+)$ or incoming $(-)$ waves asymptotically, etc. We choose $H\varphi_0=0$. A state[3] $\varphi_{n+1}^{(\pm)}$ may then be constructed out of a state $\varphi_n^{(\pm)}$ by the relation

$$\varphi_{n+1}^{(\pm)}=a_p^\dagger\varphi_n^{(\pm)}-\frac{1}{H-E_n-\omega_p\mp i\epsilon}[H_1,a_p^\dagger]\varphi_n^{(\pm)}.\tag{4}$$

Here $H\varphi_n^{(\pm)}=E_n\varphi_n^{(\pm)}$, and the state $\varphi_{n+1}^{(\pm)}$ contains in addition to the mesons of $\varphi_n^{(\pm)}$ a real meson in the state p. If we define a new operator $b_p^{(\pm)\dagger}$ by

$$b_p^{(\pm)\dagger}=a_p^\dagger-\sum_{s,t}\frac{|\varphi_t^{(\pm)}\rangle\langle\varphi_t^{(\pm)}|[H_1,a_p^\dagger]|\varphi_s^{(\pm)}\rangle\langle\varphi_s^{(\pm)}|}{E_t-E_s-\omega_p\mp i\epsilon},\tag{5}$$

we can then write

$$\varphi_{n+1}^{(\pm)}=b_p^{(\pm)\dagger}\varphi_n^{(\pm)}.\tag{6}$$

With this definition, it is straightforward to show that we also have

$$\delta_{p,n}\varphi_{n-1}^{(\pm)}=b_p^{(\pm)}\varphi_n^{(\pm)},\tag{7}$$

that is, the operator $b_p^{(\pm)}$ acting on $\varphi_n^{(\pm)}$ gives zero unless one of the n mesons in $\varphi_n^{(\pm)}$ is the meson p, in which case it gives the state with that meson missing.

The operators $b_p^{(\pm)}$ and $b_p^{(\pm)\dagger}$ may thus be interpreted as destroying and creating physical mesons in the exact scattering states of H.

The usual properties of creation and destruction operators are easily verified; namely,

$$[b_p^{(\pm)},b_q^{(\pm)\dagger}]=\delta_{pq},\tag{8}$$

and we find, as is to be expected,

$$H=\textstyle\sum_k\omega_k b_k^{(\pm)\dagger}b_k^{(\pm)}.\tag{9}$$

It remains now to relate the two kinds of operators, $b^{(+)}$ and $b^{(-)}$ corresponding to states with outgoing and ingoing boundary conditions.

Let k represent a state specified by the angular momentum l, m and the energy k; i.e., $k\leftrightarrow k, l, m$. The meson field may then be expanded in the complete set

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[1] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[2] Drell, Friedman, and Zachariasen, Phys. Rev. **104**, 236 (1956). Hereafter referred to as I.

[3] We use the notation φ_n to specify a state of n mesons, without explicit indication of the states occupied by these mesons.

of scattering states of the Hamiltonian H:

$$\varphi_\sigma(x)=\sum_{l,m,k}(2\omega_k)^{-\frac{1}{2}}\times[b_{lmk\sigma}{}^{(\pm)}u_{lk}{}^{(\pm)}(x)Y_{lm}(\Omega_x)+\text{c.c.}]. \quad (10)$$

Here the $u_{lk}{}^{(\pm)}$ are radial eigensolutions (in the case of separable H) corresponding to plane waves plus outgoing $(+)$ or ingoing $(-)$ scattered waves. Now the difference between the outgoing and ingoing radial wave functions for the scattering is just

$$u_{lk}{}^{(+)}(x)=\exp[2i\delta_l{}^{(1)}(k)]u_{lk}{}^{(-)}(x), \quad (11)$$

where $\delta_l{}^{(1)}$ is the lth phase shift produced by the interaction H_1. Thus, we have

$$b_{lmk\sigma}{}^{(+)}=\exp[-2i\delta_l{}^{(1)}(k)]b_{lmk\sigma}{}^{(-)}. \quad (12)$$

Next let an additional interaction H_2 be added to the Hamiltonian H. Write

$$H_T=H_1+H_2+H_0, \quad (13)$$

with eigenstates $\Psi_0, \Psi_k{}^{(\pm)}\cdots$. Denote by E_s the self-energy due to the presence of H_2. Thus,

$$\begin{aligned} H_T\Psi_0&=E_s\Psi_0,\\ H_T\Psi_k{}^{(\pm)}&=(E_s+\omega_k)\Psi_k{}^{(\pm)}, \end{aligned} \quad (14)$$

etc. We suppose that the problem for the Hamiltonian $H=H_1+H_0$ is exactly solved; that is, we know the states φ, and the expansion of the fields in terms of the operators $b_k{}^{(+)}$ and $b_k{}^{(-)}$. Thus we assume that expressions like $[H_2,b_k{}^{(\pm)}]$ can be evaluated. We also assume that the S matrix for H_1 alone is known.

The S matrix for scattering due to both H_1 and H_2 is

$$S_{pq}=\langle\Psi_p{}^{(-)}|\Psi_q{}^{(+)}\rangle. \quad (15)$$

Writing

$$\Psi_p{}^{(\pm)}=\varphi_p{}^{(\pm)}-\frac{1}{H-\omega_p\mp i\epsilon}(H_2-E_s)\Psi_p{}^{(\pm)}, \quad (16)$$

and performing some algebraic manipulations, we are led to

$$S_{pq}=\langle\varphi_p{}^{(-)}|\varphi_q{}^{(+)}\rangle-2\pi i\delta(\omega_p-\omega_q)\times\langle\Psi_p{}^{(-)}|(H_2-E_s)|\varphi_q{}^{(+)}\rangle. \quad (17)$$

The first term here is just the S matrix due to H_1 alone. This being presumed known, we confine our attention to the second term. Using

$$\varphi_p{}^{(\pm)}=b_p{}^{(\pm)\dagger}\varphi_0, \quad (18)$$

and

$$Z_2\varphi_0=\Psi_0+\frac{1}{H}(1-P_0)(H_2-E_s)\Psi_0, \quad (19)$$

where $Z_2=\langle\varphi_0|\Psi_0\rangle$ and P_0 projects[4] onto φ_0, we construct

$$S_{pq}=\langle\varphi_p{}^{(-)}|\varphi_q{}^{(+)}\rangle-\frac{2\pi i}{Z_2}\delta(\omega_p-\omega_q)\langle\Psi_p{}^{(-)}|[H_2,b_q{}^{(+)\dagger}]|\Psi_0\rangle. \quad (20)$$

It should be noted that the two coefficients of $2\pi i\delta(\omega_p-\omega_q)$ in Eqs. (17) and (20) are not themselves equal unless $\omega_p=\omega_q$. The renormalization constant Z_2 may be incorporated into the coupling constant of H_2, thus including renormalization effects of H_1. We therefore drop the Z_2.

We define the transition amplitude

$$\tau_q(p)=\langle\Psi_p{}^{(-)}|[H_2,b_q{}^{(+)\dagger}]|\Psi_0\rangle. \quad (21)$$

Again letting $q\,p$, etc., specify states of a given angular momentum l, we can write

$$\tau_q(p)=T_{q,l}(p,l)\exp[2i\delta_l{}^{(1)}(q)], \quad (22)$$

where

$$T_{q,l}(p,l)=\langle\Psi_p{}^{(-)}|[H_2,b_q{}^{(-)\dagger}]|\Psi_0\rangle. \quad (23)$$

T now satisfies the usual type of Chew-Low integral equation,[1,2] obeys a unitarity condition on the energy shell, and is therefore of the form $\sin\delta_l{}^{(2)}\exp[i\delta_l{}^{(2)}]$. The total scattering amplitude for a given l then takes the form

$$\sin\delta_l{}^{(1)}\exp[i\delta_l{}^{(1)}]+\exp[2i\delta_l{}^{(1)}]\sin\delta_l{}^{(2)}\exp[i\delta_l{}^{(2)}]=\sin(\delta_l{}^{(1)}+\delta_l{}^{(2)})\exp[i(\delta_l{}^{(1)}+\delta_l{}^{(2)})], \quad (24)$$

the first term on the left-hand side being the scattering due to H_1 alone.

The interaction H_1 has thus been separated out from the problem. Its presence influences the equation for scattering by H_2 alone only in that the inhomogeneous term of the Chew-Low equation now involves $[H_2,b_k{}^{(-)\dagger}]$ instead of $[H_2,a_k{}^\dagger]$.

APPLICATION

Let us now apply the above formalism to an explicit example. We consider the pair term

$$H_1=\lambda_0{}^0\int\varphi(\mathbf{x})\cdot\varphi(\mathbf{x})s(\mathbf{x})d^3\mathbf{x}, \quad (25)$$

with the source density $s(\mathbf{x})=3/4\pi a^3$, $|\mathbf{x}|<a$, and $=0$ for $|\mathbf{x}|>a$. The eigenproblem for H_0+H_1 can be solved exactly.[5,6] We write

$$\varphi_\sigma(\mathbf{x})=\sum_{l,k}(2\omega_k)^{-\frac{1}{2}}[b_{lk\sigma}{}^{(-)}u_{lk}{}^{(-)}(r)P_l(\cos\theta)+\text{c.c.}], \quad (26)$$

[4] We wish to thank Dr. S. Gartenhaus for a discussion of this point.

[5] S. D. Drell and E. M. Henley, Phys. Rev. 88, 1053 (1952).
[6] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), second edition, p. 77.

and exhibit the ingoing radial solutions for $l=0$:

$$u_{0k}^{(-)}(r)=\begin{cases}\cos\delta_0\,\mathrm{sech}\beta a i_0(\beta r), & r<a\\ \cos\delta_0\left[j_0(kr)+ka\left(1-\dfrac{\tanh\beta a}{\beta a}\right)n_0(kr)\right], & r>a\end{cases}\quad(27)$$

where

$$\beta a=\left[\frac{3\lambda_0^0}{2\pi}\left(\frac{1}{Ma}\right)-k^2a^2\right]^{\frac{1}{2}}.\quad(28)$$

With this expansion,

$$H=\sum_{\sigma kl}\omega_k b_{lk\sigma}^{(-)\dagger}b_{lk\sigma}^{(-)},\quad(29)$$

so the $b_{lk}^{(-)}$ above are the required operators.

S-wave scattering due to this H_1 alone is given by

$$\tan\delta_0(k)=-ka\left(1-\frac{\tanh\beta a}{\beta a}\right).\quad(30)$$

Now we include a second coupling term

$$H_2=\lambda^0\boldsymbol{\tau}\cdot\left\{\int\boldsymbol{\varphi}(\mathbf{x})s(\mathbf{x})d^3\mathbf{x}\right\}\times\left\{\int\boldsymbol{\pi}(\mathbf{x})s(\mathbf{x})d^3\mathbf{x}\right\}.\quad(31)$$

The inhomogeneous term in the Chew-Low equation for the transition amplitude [Eq. (23)] describing S-wave scattering from H_2 is

$$\langle\Psi_0|[b_p^{(-)},[H_2,b_q^{(-)\dagger}]]|\Psi_0\rangle.$$

This term is the same as it would be in the absence of H_1, except multiplied by r_s^2, with

$$r_s=\int_0^\infty s(r)u_0^{(-)}(kr)r^2dr\bigg/\int_0^\infty s(r)j_0(kr)r^2dr.\quad(32)$$

So far we have been guided by simplicity in considering a source density[5] which is a square cutoff in coordinate space. However, in the papers of Chew and Low, and in I, a square cutoff in momentum space is used. For comparison we have computed r_s both with a square cutoff and with a Gaussian cutoff, and have obtained very similar answers. We thus feel that the detailed shape of the cutoff may be safely ignored. Also we may treat r_s as a constant independent of energy to a good approximation for $ka\lesssim 1$.

The equation for T, then, is identical with Eq. (24) of I, with $\lambda_0=0$ and with $v(k)$ multiplied by r_s. Solutions to this equation give results quite consistent with those obtained[7] in I. Using the values for the (renormalized) constants from I, namely $\lambda_0=0.4/\mu$; $\lambda=0.4/\mu^2$, we have $r_s=0.85$.

Photoproduction with the couplings H_1+H_2 may also be discussed on the same basis. The inhomogeneous term is again multiplied by r_s, so the effective coupling constant for S-wave photoproduction is $r_s f$. In the P-wave scattering, however, the effective coupling constant becomes $r_p f$, where

$$r_p=\int_0^\infty s(r)u_1^{(-)}(kr)r^2dr\bigg/\int_0^\infty s(r)j_1(kr)r^2dr\approx 1.0.$$

If not significant, it is nevertheless amusing to note that the agreement between the coupling constant f obtained from photoproduction and from P-wave scattering is improved.[8]

The proof of the Kroll-Ruderman theorem may be carried through just as in I, using b's instead of a's.

As far as pion pair production is concerned, the following comments may be made:

(i) The theorem[9] relating pair production to P-wave scattering is valid only in the absence of S-wave scattering. The presence of H_1 in the unperturbed Hamiltonian therefore destroys this theorem.

(ii) The results of Bincer[10] are not altered by this type of treatment of the H_1 term.

(iii) An exact calculation of pion pairs due to H_1 alone is possible using the methods of reference 5. The contribution to one S-wave and one P-wave pion is down from the perturbation result of Lawson[11] by a factor

$$\left[\frac{Ma(1-\tanh\beta a/\beta a)}{(2M/\mu)^2(g^2/4\pi)}\right]^2\approx 4\times 10^{-4},$$

corresponding to the parameters in I, showing the effect of H_1 alone to be negligible.[12]

These methods also lend themselves naturally to the inclusion of Coulomb effects in the analysis of pion-nucleon scattering.

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[7] Notice that the pair term, Eq. (5) is here treated as a local interaction in contrast with I, where separability was assumed. For simplicity we still maintain the separability assumption in Eq. (6).

[8] See discussion below Eq. (67) in I.

[9] R. E. Cutkosky and F. Zachariasen, Phys. Rev. **103**, 1108 (1956).

[10] A. M. Bincer, Phys. Rev. **105**, 1399 (1957), this issue.

[11] R. D. Lawson, Phys. Rev. **92**, 1272 (1953). In this notation $g^2/4\pi$ corresponds to $f^2\approx 0.07$.

[12] A large damping of the pair production due to a φ^2 coupling has also been demonstrated by A. Petermann, Phys. Rev. **103**, 1053 (1956).